A REVIEW ON THE LEBESGUE SPACES

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Abstract

In this article, we begin with classical Lebesgue spaces L^p with p being constant and review the various properties such as completeness and duality of the space. To this end, we also discuss the boundedness of Hardy-Littlewood maximal function and interpolation on such spaces. Finally, we focus our attention on variable exponent Lebesgue spaces and review various results on it. Moreover, we also see the differences in between these Lebesgue spaces.

Keywords: Lebesgue space, interpolation, Hardy-Littlewood maximal function, duality.

1. Introduction

In mathematics, L^p spaces, are generated using a natural generalization of the p-norm for finite-dimensional vector spaces. They are also popularly known as Lebesgue spaces, which is named after Henri Lebesgue. L^p spaces form an important class of Banach spaces in functional analysis. Moreover, they also form of topological vector spaces. They have very important roles in the mathematical analysis of measure and probability spaces. Thus, the Lebesgue spaces are used also in the theoretical discussion of problems in physics, statistics, finance, engineering, and other disciplines. One of the nice property of these Lebesgue spaces is that after the number of natural operations on Lebesgue spaces, it again forms the same space. There are number of types of these Lebesgue spaces such as classical Lebesgue space L^p, with p being a constant, weak Lebesgue spaces, weighted Lebesgue spaces and variable exponent Lebesgue spaces $L^{p(.)}$ with $p(.)$ begin variable. The variable exponent Lebesgue space is considered as the most general type of Lebesgue space. In this article, we first begin with classical Lebesgue space which is the simplest type of space on all such spaces. We review the various properties on classical Lebesgue space such as completeness, separability, duality, interpolation and boundedness of some maximal functions on these Lebesgue spaces.

We will focus more on variable exponent Lebesgue space in which we review the various properties which were discussed in the classical case and see how these properties pass from a simple to general space. We begin with classical Lebesgue space:

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1.1. Classical Lebesgue Space:

Lebesgue space L^p with $1 \leq p \leq \infty$ is known as classical Lebesgue space. We first recall the definition:

Definition: Let $1 \leq p < \infty$. The Lebesgue space defined on a set Ω, denoted by $L^p(\Omega)$, is the space of all equivalent class of measurable functions $f: \Omega \to \mathbb{R}$ for which $\|f\|_{L^p(\Omega)}$ is finite where

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p \, d\mu \right)^{\frac{1}{p}}.$$

Moreover, when $p = \infty$, we define

$$\|f\|_{L^p(\Omega)} = ess \sup_{x \in \Omega} |f(x)|.$$

For any two measurable functions f, g and α being scalars, we have for $1 \leq p \leq \infty$

a. $\|f\|_{L^p(\Omega)} \geq 0$
b. $\|f\|_{L^p(\Omega)} = 0$ if and only if $f = 0$ a.e.
c. $\|\alpha f\|_{L^p(\Omega)} = |\alpha| \|f\|_{L^p(\Omega)}$.
d. $\|f + g\|_{L^p(\Omega)} \leq \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)}$

This shows that under a.e. equal setting, $\|.\|_{L^p(\Omega)}$ defines a norm and hence $L^p(\Omega)$ is normed space. In addition to this, under this norm, the classical Lebesgue space is a complete normed space as given by the Riesz-Fischer theorem and is stated as follows:

Theorem [Riesz-Fischer]: Let Ω be a measurable set and $1 \leq p \leq \infty$. Then $L^p(\Omega)$ is a complete normed space i.e. $L^p(\Omega)$ is a Banach space.

Moreover, $L^p(\Omega)$ is also a separable space as given by the following theorem:

Theorem: Let Ω be a measurable set and $1 \leq p \leq \infty$. Then $L^p(\Omega)$ is separable.

For the classical Lebesgue space, we next state the result on the duality of the space:

Theorem: Let $1 \leq p < \infty$ and q be its conjugate index such that $\frac{1}{p} + \frac{1}{q} = 1$. Then the dual space of L^p, denoted by $(L^p)^*$, is the space L^q in the following sense:
For all bounded linear functional h on L^p there is an unique function $f \in L^q$ such that

$$h(g) = \int_\Omega g(x)f(x)d\mu(x)$$

for all $g \in L^q$. Also $\|h\|_{(L^p)^*} = \|f\|_{L^q}$.

We note that the above result is not true when $p = \infty$.Readers are suggested to refer Dienng et al. (2005) for the proof of the above theorems.

We next continue our discussion of classical Lebesgue space in connection of boundedness of Hardy-Littlewood maximal function. So let us consider the classical Lebesgue space $L^p(\Omega)$. In order to do so, we first recall the definition of Hardy-Littlewood maximal function.

Definition: Let $f \in L^1_{loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. Then the Hardy-Littlewood maximal function associated to f is denoted by Mf and is defined as

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy$$

where the supremum is taken over all balls $B(x,r)$ centered at the point x and radius r. Therefore, this is centered Hardy-Littlewood maximal function.

Similarly, we can define uncentered Hardy-Littlewood maximal associated to f, denoted by $M^u f$, is defined as:

$$M^u f(x) = \sup_{r>0, x\in B} \frac{1}{|B|} \int_B |f(y)|dy$$

where the supremum is taken over all balls B containing x. These two definitions are equivalent as we can show that $Mf(x) \leq M^u f(x) \leq 2^n Mf(x)$ and hence we do not differentiate between these two forms. We now state a result on boundedness of Hardy-Littlewood Maximal function on $L^p(\mathbb{R}^n)$.

Theorem: (Hardy-Littlewood, 1930) Let f be a measurable function on $\mathbb{R}^n$.
Then

a) If $f \in L^p(\mathbb{R}^n)$, for $1 \leq p \leq \infty$, then Mf is finite a.e.
b) If $f \in L^1(\mathbb{R}^n)$, then for all $\lambda > 0$, we have

$$|\{x \in \mathbb{R}^n : Mf(x) > \lambda\}| \leq \frac{3^n}{\lambda} \int_{\mathbb{R}^n} |f|dx.$$
$$= \frac{3^n}{\lambda} \|f\|_{L^1(\mathbb{R}^n)}$$

c) If $f \in L^p(\mathbb{R}^n), 1 < p \leq \infty$, then $Mf \in L^p(\mathbb{R}^n)$ and $\|Mf\|_p \leq A_p \|f\|_p$ where A_p is constant depending only on p and n.

Some remarks are in order:

i. (b) says that Hardy-Littlewood maximal operator M is of weak-type $(1, 1)$. We note that (c) is not true for $p = 1$. In fact, one can show that $\|Mf\|_1 = \infty$ whenever $f \in L^1(\mathbb{R}^n)$ with f identically zero. As an example we can take $f(x) = \chi_A$ where $A = [-1, 1]$. For this characteristic function, we have $f \in L^1(\mathbb{R})$ but the maximal function $Mf \notin L^1(\mathbb{R})$. Also note that (a) and (b) are obvious if $p = \infty$.
ii. (c) says that the M is of strong type (p, p) for $1 < p \leq \infty$. Consequently, M is of weak type (p, p) for $1 < p < \infty$. This follows simply using the Chebychev's inequality and strong type (p, p) condition.

After discussing the boundedness of H-L maximal function, we next discuss the interpolation spaces of Lebesgue spaces. We consider the situation on the interpolation spaces in between Lebesgue spaces L^1 and L^p with $1 < p < \infty$. For this we first recall some basic definitions from Astashkin & Maligranda (2003).

Let x^* denote the nonincreasing arrangement of $|x|$ and $\|.\|_X$ denote the norm on a space X. A Banach space X of measurable functions on the interval $[0, 1]$ is said to be symmetric if $y \in X$ and $x^*(t) \leq$

$y^*(t)$ for $t \in [0,1]$ imply that $x \in X$ and $\|x\|_X \le \|y\|_Y$.

A symmetric space X is said to satisfy Fatou property if for any sequence $\{x_n\}$ in X with $x_n \ge 0, x_n \nearrow x$ and $\sup_n \|x_n\|_X < \infty$, we have $x \in X$ and corresponding norm sequence $\|x_n\|_X \nearrow \|x\|_X$.

The symmetric space X is said to satisfy absolutely continuous norm if for any arbitrary $x \in X$ and any sequence $\{x_n\}$ of measurable functions on $[0,1]$ such that $|x| \ge x_n \ge 0$ and $x_n \downarrow 0$ imply that $\|x_n\|_X \to 0$. Moreover, a symmetric space X is separable iff X possesses an absolutely continuous norm. We now discuss results on interpolation spaces in Lebesgue spaces. Boyd (1967) proved that:

Theorem: Let X be a symmetric space of measurable functions defined on $[0,1]$. If Boyd indices of the space X satisfy the inequalities

$$\frac{1}{q} < \alpha(X) \le \beta(X) < \frac{1}{p}$$

and X satifies Fatou property then X is an interpolation space between the spaces L_p and L_q.

This theorem shows that any bounded linear operator T is in the space L_p and L_q is also bounded in the space X. Malingranda (Maligranda, 1981) showed that the condition in Boyd's theorem in the case $q = \infty$ can be weakened. Precisely, the space X is an ineterpolation space between the Lebesuge spaces L_1 and L_∞ can be proved using only the one-side of Boyd's estimate i.e. $\beta(X) < \frac{1}{p}, 1 \le p < \infty$.

To this end, we now state the most general results on interpolation spaces of Lebesgue space.

Theorem (Astashkin & Malingranda, 2003) Suppose that $1 < p < \infty$. If a symmetric space X is an interpolation space between L_1 and L_∞ and the Boyd indices $\alpha(X) > \frac{1}{p}$, then the space X is also an interpolation space between L_1 and L_p.

The condition that X is an interpolation space beteween L_1 and L_∞ in the above theorem is unavoidable. Please see Astashkin & Maligranda (2003) for the counter example. Astashkin and Malingranda have also generalized their result and their result is:

Theorem (Astashkin & Malingranda, 2003) Suppose that $1 \le r < p < \infty$. If X is an interpolation space between the L_r and L_∞ and the Boyd indices $\alpha(X) > \frac{1}{p}$, then X is an interpolation space between L_r and L_p.

Besides classical Lebesgue space, there are other types of Lebesgue spaces such as weak type Lebesgue spaces and weighted Lebesgue spaces. We simply state their definitions.

Weak type Lebesgue spaces: Let f be a measurable function from $\mathbb{R}^n$ to set of complex numbers and $\Omega \subseteq \mathbb{R}^n$. For $0 < p < \infty$, weak type Lebesgue space, denoted by $L^p_{weak}(\Omega)$, is the set of all measurable function f defined on Ω for which:

$$\|f\|_{L^p_{weak}(\Omega)} = \sup_{t>0} t\, |\{x \in \Omega: |f(x)| > t\}|^{\frac{1}{p}} < \infty$$

for all $t \ge 0$. One can see that the classical Lebesgue space is a subset of weak type Lebesgue space which simply follows by the application of Chebychev inequality.

Weighted Lebesgue spaces: By a weight w, we mean a non-negative locally integrable function defined on the given set Ω. Let w be such a weight function on Ω and $1 \le p < \infty$. Then the weighted Lebesgue space on Ω, denoted by $L^p_w(\Omega)$, is the set of all measurable function f defined on Ω for which

$$\|f\|_{L^p_w(\Omega)} = \left(\int_\Omega |f(x)|^p\, w(x)dx\right)^{\frac{1}{p}} < \infty.$$

If we take the weight function $w(x) = 1$, then the weighted Lebesgue space is same as the classical Lebesgue space. Finally we discuss about the variable exponent Lebesgue space which is the central part of this review paper.

1.2. Variable Exponent Lebesgue Spaces:

Wladyslaw Orlicz was the first person to start the theory of variable exponent Lebesgue spaces in the year 1930 (Nguyen, 2011). He introduced the space as a special case of some other spaces. His theory of variable exponent space was studied and analyzed through at the end of the century. Later, several results and their applications made the mathematicians interested in such Lebesgue spaces.

One can see that with very few assumptions on the variable exponent function, many of the classical structure and density theorems are valid in the variable case. Moreover, these results were simply obtained by the use of well-established methods and this methodology set the standard for the first part of the decade. During this time many positive results on the variable exponent Lebesgue spaces was established. Even though one can find many nontrivial situations in which one cannot hope to extend a result known for a classical Lebesgue space to the variable exponent Lebesgue case.

We first begin with the definition of exponent Lebesgue space:

Definition: Let $\Omega \subset \mathbb{R}^n$. Then the classical $L^p, 1 \leq p < \infty$ space is the collection of all measurable functions f for which the integral $\int_\Omega |f|^p d\mu < \infty$ with μ being the underlying measure of the space. Here exponent p is a constant. Now let us replace the constant p by a variable exponent $p(x)$ such that $p(x): \Omega \to [1, \infty]$ is a measurable function. Then collection of measurable functions f such that the integral $\int_\Omega |f|^{p(x)} d\mu < \infty$ forms a new Lebesgue space, known as variable exponent L^p spaces. There are many approaches to define this space and we define it from Izuki et al. (2014).

Let f be a complex-valued measurable function defined on $\Omega \subset \mathbb{R}^n$ and $p: \Omega \to [1, \infty]$ be measurable function. Then a functional ρ_p defined on the space of complex valued measurable function f is given by

$$\rho_p(f) = \int_{\Omega \setminus \Omega_\infty} |f(x)|^{p(x)} d\mu(x) + \|f\|_{L^\infty(\Omega_\infty)}$$

where $\Omega_\infty = \{x \in \Omega: p(x) = \infty\}$. This functional ρ_p is called the $p-$modular for the given space.

Thus, the variable exponent Lebesgue space, denoted by $L^{p(.)}(\Omega)$, is the collection of all measurable function defined on Ω for which the p-modular $\rho_p(f/t_0)$ is finite for some $t_0 > 0$. For the norm on this space, we consider a functional given by:

$$\|f\|_{L^{p(.)}(\Omega)} = \inf\left\{t > 0: \rho_p\left(\frac{f}{t}\right) \leq 1\right\} - - - -(1)$$

Clearly, we have

i. $\|f\|_{L^{p(.)}(\Omega)} \geq 0,$
ii. $\|f\|_{L^{p(.)}(\Omega)} = 0$ if and only if $f = 0$ a.e.
iii. $\|\alpha f\|_{L^{p(.)}(\Omega)} = |\alpha| \, \|f\|_{L^{p(.)}(\Omega)}$.

Moreover, the triangle inequality

$$\|f + g\|_{L^{p(.)}(\Omega)} \leq \|f\|_{L^{p(.)}(\Omega)} + \|g\|_{L^{p(.)}(\Omega)}$$

also holds (Izuki et al., 2011).

This shows that the functional given by (1) is a norm under the almost everywhere equality. We now discuss various results that hold in the variable exponent Lebesgue space. We begin with Holder's inequality. For the exponent $p(.): \Omega \to [1, \infty]$, we define conjugate index $p'(.): \Omega \to [1, \infty]$ as

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$$

with $x \in \mathbb{R}^n$.

Holder's inequality: Let $p(.): \Omega \to [1, \infty]$ and $p'(.): \Omega \to [1, \infty]$ be such that $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. Then for all $f_1 \in L^{p(.)}(\Omega)$ and $f_2 \in L^{p'(.)}(\Omega)$, we have

$$\int_\Omega |f_1(x) f_2(x)| dx \leq r(p) \|f_1\|_{L^{p(.)}(\Omega)} . \|f_2\|_{L^{p'(.)}(\Omega)}$$

where $r(p) = 1 + \frac{1}{p^-} + \frac{1}{p^+}$ with $p^- = ess \inf_{x \in \Omega} p(x)$ and $p^+ = ess \sup_{x \in \Omega} p(x)$. For the proof, please refer Izuki et al. (2014).

Theorem [Completeness]: Let $p(.): \Omega \to [1, \infty]$ be a measurable function. Then the variable exponent Lebesgue space $L^{p(.)}(\Omega)$ with the norm $\|.\|_{L^{p(.)}(\Omega)}$ is a complete normed space, i.e., a Banach space.

For the proof of the above result, as usual we take a Cauchy sequence and show that it converges to a function in the space (Izuki et al., 2014).

Moreover, the dual space of $L^{p(.)}(\Omega)$ with $p(.): \Omega \to [1, \infty]$ being variable exponent, is denoted by $L^{p(.)}(\Omega)^*$ and is defined as:

$$L^{p(.)}(\Omega)^* = \{T: L^{p(.)}(\Omega) \to \mathbb{C}: T \text{ is linear and bounded}\}.$$

The norm on this space is given by:

$$\|T\|_{L^{p(.)}(\Omega)} = \sup\{|T(u)|: \|u\|_{L^{p(.)}(\Omega)} \leq 1\}.$$

In connection with the dual space, we have following theorem. For the proof, see Izuki et al. (2014).

Theorem: Let $p'(.)$ denote the conjugate index of the variable exponent $p(.): \Omega \to \infty$. For a measurable function $f \in L^{p'(.)}(\Omega)$, define a functional T_f given by:

$$T_f(u) = \int_{\Omega} f(x)u(x)dx$$

with $u \in L^{p(.)}(\Omega)$. Then the integral $T_f u$ converges absolutely. Moreover, the functional $T_f \in L^{p(.)}(\Omega)^*$ and we have

$$\frac{1}{3}\|f\|_{L^{p'(.)}(\Omega)} \leq \|T_f\|_{L^{p(.)}(\Omega)^*} \leq \left(1 + \frac{1}{p^-} - \frac{1}{p^+}\right)\frac{1}{3}\|f\|_{L^{p'(.)}(\Omega)}.$$

From the above theorem, one can conclude that $L^{p'(.)}(\Omega) \subset L^{p(.)}(\Omega)^*$. This shows that in some sense, the space $L^{p'(.)}(\Omega)$ can be identified with the dual space of variable exponent Lebesgue space. In the case of $p^+ < \infty$, we have a definite result on the dual space and is given by the following theorem from Izuki et al. (2014).

Theorem: Let $p^+ = ess \sup_{x \in \Omega} p(x) < \infty$ with $p(.): \Omega \to [1, \infty]$ being variable exponent. Then for all linear functional $F \in L^{p(.)}(\Omega)^*$, there exists an unique element
$f \in L^{p'(.)}(\Omega)$ such that for $u \in L^{p(.)}(\Omega)$

$$F(u) = \int_{\Omega} f(x)u(x)dx.$$

In addition to this, we have

$$\frac{1}{3}\|f\|_{L^{p'(.)}(\Omega)} \leq \|F\|_{L^{p(.)}(\Omega)^*} \leq \left(1 + \frac{1}{p^-} - \frac{1}{p^+}\right)\|f\|_{L^{p'(.)}(\Omega)}.$$

The above theorem shows that, the space $L^{p(.)}(\Omega)^*$ is a subset of space $L^{p'(.)}(\Omega)$.

For the boundedness condition for Hardy-Littlewood maximal function defined on the variable exponent Lebesgue space, we first discuss the Log-Holder condition. For a variable exponent $p(.): \Omega \to [1, \infty]$, the condition given by

$$|p(x) - p(y)| \leq \frac{c_*}{\log(1/|x-y|)} ------ (1)$$

for $|x - y| \leq \frac{1}{2}$ for $x, y \in \mathbb{R}^n$ is called local log-Holder continuity condition. Moreover,

$$|p(x) - p_\infty| \leq \frac{c^*}{\log(e + |x|)} ---- (2)$$

for $x \in \mathbb{R}^n$ is called log- Holder decay condition at infinity. Here c_*, c^* are positive constants and are independent of x and y. The boundedness of Hardy-Littlewood maximal operator on the variable exponent Lebesgue space is guaranteed by the following theorem proved in Izuki et al. (2014).

Theorem: Let $p(.)$ is a measurable function defined from $\mathbb{R}^n$ to $[1, \infty]$. If $1 < p^- \leq p^+ < \infty$ and variable exponent $p(.)$ satisfies both local log-Holder condition and log-Holder decay condition at infinity, then the Hardy-Littlewood maximal function defined the variable exponent Lebesgue space is bounded on this space i.e. $Mf \in B\left(L^{p(.)}(\mathbb{R}^n)\right)$.

At the end, we see how the variable exponent Lebesgue space is fundamentally different from the classical Lebesgue space. We consider examples from Nguyen (2011).

Let us take $\Omega = [1, \infty)$ and $d\mu = dx$ and $p(x) = x$. Let α be a nonzero complex number which denotes the constant function $x \mapsto \alpha$. For $t > |\alpha|$, one can show that

$$\int_1^\infty \left|\frac{\alpha}{t}\right|^x dx = \lim_{n \to \infty} \frac{(|\alpha|t^{-1})^n - |\alpha|t^{-1}}{\ln(|\alpha|t^{-1})}.$$

This shows that $\alpha \in L^x$. This is a contrast to the fact that only constant function in the classical Lebesgue space on $[1, \infty)$ is the zero function. Next if $\Omega = \mathbb{R}^n$ and $d\mu = dx$. In this setting the translation invariance property of classical Lebesgue space is obvious. But in the variable exponent setting, there exists a nonzero translation that is not continuous on the $L^{p(.)}(\mathbb{R}^n)$. These examples motivate the fact that for what extent the results on the classical Lebesgue space can be extended to variable exponent case. Finally, readers are suggested to refer to a paper by Dienng et al. (2005) for some open problems in the variable exponent Lebesgue spaces.

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